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Formulas for A_n- and B_n-solutions of WDVV equations

S.M. Natanzon*

Independent University of Moscow, Moscow, Russia
Moscow State University, 119899 Moscow, Russia

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Abstract

The simplest non-trivial solutions of WDVV equations are A_n-and B_n-potentials, which describe metrics of Saito on spaces of versal deformation of A_n-and B_n-singularities. These are some polynomials, which were known for $n \leq 4$. In this paper, we find the potentials for all A_n-and B_n-singularities. We give a recurrence formula for coefficients of KP and n-KdV hierarchy.
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1. Introduction

WDVV equations appeared in papers of Witten [9], Dijkgraaf et al. [5] as equations for primary free energy of two-dimensional topological field theory. The simplest non-trivial solutions of WDVV equations are A_n-and B_n-potentials, which describe metrics of Saito on spaces of versal deformation of A_n-and B_n-singularities. According to Witten [10], the coefficients of A_n-potentials are intersection numbers of Mumford–Morita–Miller classes of a moduli space of spheres with punctures and n-spin structures.

The A_n-potential has the following description [1]. Let us consider the space of polynomials $M_n = \{z^{n+1} + a_1 z^{n-1} + \cdots + a_n | a_i \in \mathbb{C}\}$. The tangent space T_p of the point $p = p(z) \in M$ is a set of polynomials $Q_n = \{b_1 z^{n-1} + \cdots + b_n | b_i \in \mathbb{C}\}$. On T_p define a structure of a Frobenius algebra, considering that $v_1 * v_2 = v_1 \cdot v_2 \,(\mathrm{mod}\, \mathrm{d}p/\mathrm{d}z)$ is multiplication of the algebra and $g(v_1, v_2) = \mathrm{Res}_{z=\infty}(v_1 \cdot v_2)/(\mathrm{d}p/\mathrm{d}z)$ is its scalar product. Then M_n

* Present address: Moscow State University, 119899 Moscow, Russia.
E-mail address: natanzon@mccme.ru (S.M. Natanzon).

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is a Frobenius manifold. It has the flat coordinates $(t^1, \dots, t^n)$ such that $g(\partial/\partial t^i, \partial/\partial t^j) = \delta_{i+j,n+1}$ and $E(t^i) = ((n+2-i)/(n+2))t^i$, where the vector field E is given by $E = \sum_{k=1}^{n}((k+1)/(n+1))a_k(\partial/\partial a^k)$ in coordinates $\{a_1, \dots, a_n\}$. Moreover, there exists a polynomial F such that $(\partial/\partial t^i) * (\partial/\partial t^j) = \sum_{k=1}^{n}(\partial^3 F/\partial t^i \partial t^j \partial t^{n+1-k})(\partial/\partial t^k)$. This polynomial is called A_n-potential.

According to [5], A_n-potential is the string solution of n-KdV (Gelfand–Dikii) hierarchy. According to [6], A_n-potential is the string solution of dispersionless (quasi-classical) limit of n-KdV hierarchy. The coincidence of these solutions follows directly from our representation of A_n-potential. Moreover, we give recurrence formulas for calculation of all these polynomials. Hitherto A_n-potential was known only for $n \leq 4$ [1,2,10]. We also give recurrence formulas for coefficients of KP and n-KdV hierarchy. Results of Zuber [11] give a reduction of B_n-potentials to A_n-potentials.

We formulate the main result of the paper in Section 2. In Section 3, we prove auxiliary combinatorial lemma. It is used in Section 4 for representation of KP hierarchy as an equation on a Baker–Akhiezer function. We use a representation of the Baker–Akhiezer function by τ-function and describe KP hierarchy as an infinite system of differential equations on $v = -\log \tau$ in Section 5. We give an analogue of this representation for n-KdV hierarchy in Section 6. Using this representation we prove the main theorem in Section 7.

2. Main theorems

For calculation of the A_{n-1}-potential, we use combinatorial coefficients $P_n(i_1 \cdots i_m)$, where n and s_j are natural numbers. Define

$$P_n(i) = n, \qquad P_n(s_1 \cdots s_m) = C_n^m - \sum_{q=1}^{m-1} P_n(s_1 \cdots s_q) C^{m-q}_{n-q-(s_1+\cdots+s_q)},$$

where

$$0! = 1, \quad C_p^t = \frac{p!}{t!(p-t)!} \text{ for } p \geq t \geq 0 \text{ and } C_p^t = 0 \text{ in other cases.}$$

Let us assume $x_0 = 0$. For $q > 0$, we recursively define polynomials $x_{-q}(x_1, \dots, x_{n-1})$ putting

$$x_{-q} = -\frac{1}{n}\sum_{m=2}^{\infty}\sum P_n(s_1 \cdots s_m) x_{n-s_1} \cdots x_{n-s_m},$$

where the second sum is taken over all natural numbers $(s_1, \dots, s_m)$ such that $\sum_{i=1}^{m} s_i = q + n + 1 - m$.

This polynomial has the form

$$x_{-q} = \sum_{m=1}^{\infty}\sum B_{qi_1 \cdots i_m} x_{i_1} \cdots x_{i_m},$$

where $B_{qi_1,\dots,i_m}$ are constants and the second sum is taken over numbers $(i_1, \dots, i_m)$ such that $1 \le i_1 \le \cdots \le i_m \le n-1$. Define

$$A_{i_1\cdots i_m} = -\frac{1}{i_1 p} B_{i_1\cdots i_m},$$

where p is the number of j such that $i_j = i_1$. Consider

$$F_A = \sum_{m=3}^{\infty} \sum A_{i_1\cdots i_m} x_{i_1} \cdots x_{i_m}, \qquad F_B = \sum_{m=3}^{\infty} \sum A_{2i_1-1\cdots 2i_m-1} x_{i_1} \cdots x_{i_m},$$

where the second sum is taken over all $1 \le i_1 \le \cdots \le i_m \le n-1$.

Theorem 1. *The polynomial F_A is an A_{n-1}-potential. The polynomial F_B is a B_{n-1}-potential. They satisfy the equations*

$$\sum_{\gamma=1}^{n-1} \frac{\partial^3 F}{\partial x_\alpha \partial x_\beta \partial x_\gamma} \frac{\partial^3 F}{\partial x_{n-\gamma} \partial x_\eta \partial x_\xi} = \sum_{\gamma=1}^{n-1} \frac{\partial^3 F}{\partial x_\xi \partial x_\beta \partial x_\gamma} \frac{\partial^3 F}{\partial x_{n-\gamma} \partial x_\eta \partial x_\alpha} \quad (\alpha, \beta, \eta, \xi = 1, \ldots, n-1),$$

$$\frac{\partial^3 F}{\partial x_1 \partial x_\alpha \partial x_\beta} = \delta_{\alpha+\beta,n}, \qquad \sum_{j=1}^{n-1} d_i \frac{\partial F}{\partial x_i} = \mathrm{d}F,$$

where $d_i = 1 + (1-i)/n$, $d = 2 + 2/n$ for $F = F_A$ and $d_i = 1 + (1-i)/(n-1)$, $d = 2 + 1/(n-1)$ for $F = F_B$.

3. Combinatorial lemma

For natural $s, i_1, \ldots, i_n$ and integer non-negative $j_1, \ldots, j_n$ we define

$$P_s \begin{pmatrix} i_1 & \cdots & i_n \\ j_1 & \cdots & j_n \end{pmatrix}$$

by the recurrence formula:

(1) $$P_s \begin{pmatrix} i_1 & \cdots & i_n \\ 0 & \cdots & 0 \end{pmatrix} = 0,$$

(2) $$P_s \begin{pmatrix} i \\ j \end{pmatrix} = C_s^j \quad \text{for } j > 0,$$

(3) $$P_s \begin{pmatrix} i_1 & \cdots & i_n \\ j_1 & \cdots & j_n \end{pmatrix} = \frac{1}{n!} C_s^{j_1+\cdots+j_n} \frac{(j_1+\cdots+j_n)!}{j_1!\cdots j_n!} - \sum_{q=1}^{n-1} P_s \begin{pmatrix} i_1 & \cdots & i_q \\ j_1 & \cdots & j_q \end{pmatrix} \frac{1}{(n-q)!} C_{s-(i_1+\cdots+i_q+j_1+\cdots+j_q)}^{j_{q+1}+\cdots+j_n} \frac{(j_{q+1}+\cdots+j_n)!}{j_{q+1}!\cdots j_n!}$$

for $(j_1, \dots, j_n) \neq (0, \dots, 0)$.

Let

$$\begin{bmatrix} i_1 & \cdots & i_n \\ j_1 & \cdots & j_n \end{bmatrix}$$

be the set of all matrices that appear from

$$\begin{pmatrix} i_1 & \cdots & i_n \\ j_1 & \cdots & j_n \end{pmatrix}$$

by permutation of columns. Let

$$\left\| \begin{matrix} i_1 & \cdots & i_n \\ j_1 & \cdots & j_n \end{matrix} \right\|$$

be the number of such matrices. Define

$$P_s \begin{bmatrix} i_1 & \cdots & i_n \\ j_1 & \cdots & j_n \end{bmatrix} = \sum P_s \begin{pmatrix} a_1 & \cdots & a_n \\ b_1 & \cdots & b_n \end{pmatrix},$$

where the sum is taken by all

$$\begin{pmatrix} a_1 & \cdots & a_n \\ b_1 & \cdots & b_n \end{pmatrix} \in \begin{bmatrix} i_1 & \cdots & i_n \\ j_1 & \cdots & j_n \end{bmatrix}.$$

Lemma 1. *Let $m > 0, k > 0$ and $j_n \geq 1$ for $n \leq m$. Then*

$$P_s \begin{bmatrix} i_1 & \cdots & i_m & i_{m+1} & \cdots & i_{m+k} \\ j_1 & \cdots & j_m & 0 & \cdots & 0 \end{bmatrix} = \begin{cases} 0 & \text{if } s \geq i_1 + \cdots + i_m + j_1 + \cdots + j_m, \\ \dfrac{1}{k!} \left\| \begin{matrix} i_{m+1} & \cdots & i_{m+k} \\ 0 & \cdots & 0 \end{matrix} \right\| P_s \begin{bmatrix} i_1 & \cdots & i_m \\ j_1 & \cdots & j_m \end{bmatrix} & \text{if } s < i_1 + \cdots + i_m + j_1 + \cdots + j_m. \end{cases}$$

Proof. Prove at first the lemma for $m = 1$ using induction by k. For $m = k = 1$,

$$\begin{aligned} P_s \begin{bmatrix} i_1 & i_2 \\ j_1 & 0 \end{bmatrix} &= P_s \begin{pmatrix} i_1 & i_2 \\ j_1 & 0 \end{pmatrix} + P_s \begin{pmatrix} i_2 & i_1 \\ 0 & j_1 \end{pmatrix} \\ &= \frac{1}{2} C_s^{j_1} - P_s \begin{pmatrix} i_1 \\ j_1 \end{pmatrix} C^0_{s-(i_1+j_1)} + \frac{1}{2} C_s^{j_1} - P_s \begin{pmatrix} i_2 \\ 0 \end{pmatrix} C^0_{s-i_2} \\ &= C_s^{j_1} - C_s^{j_1} C^0_{s-(i_1+j_1)} = \begin{cases} 0 & \text{if } s \geq i_1 + j_1, \\ C_s^{j_1} = P_s \begin{bmatrix} i_1 \\ j_1 \end{bmatrix} & \text{if } s < i_1 + j_1. \end{cases} \end{aligned}$$

To prove the lemma for $m = 1, k = N$, we assume that it is proved for $m = 1, k < N$. If $s \geq i_1 + j_1$, then

$$P_1\begin{bmatrix} i_1 & i_2 & \cdots & i_{k+1} \\ j_1 & 0 & \cdots & 0 \end{bmatrix} = \frac{1}{k!}C_s^{j_1}\begin{Vmatrix} i_2 & \cdots & i_{k+1} \\ 0 & \cdots & 0 \end{Vmatrix} - P_s\begin{pmatrix} i_1 \\ j_1 \end{pmatrix} C^0_{s-(i_1+j_1)}\frac{1}{k!}\begin{Vmatrix} i_2 & \cdots & i_{k+1} \\ 0 & \cdots & 0 \end{Vmatrix} = 0.$$

If $s < i_1 + j_1$, then

$$P_1\begin{bmatrix} i_1 & i_2 & \cdots & i_{k+1} \\ j_1 & 0 & \cdots & 0 \end{bmatrix} = \frac{1}{k!}C_s^{j_1}\begin{Vmatrix} i_2 & \cdots & i_{k+1} \\ 0 & \cdots & 0 \end{Vmatrix} - AC^0_{s-(i_1+j_1)} = \frac{1}{k!}\begin{Vmatrix} i_2 & \cdots & i_{k+1} \\ 0 & \cdots & 0 \end{Vmatrix} P_s\begin{bmatrix} i_1 \\ j_1 \end{bmatrix}.$$

Thus the lemma is proved for $m = 1$.

To prove the lemma for $m = N$, we assume that it is proved for $m < N$. Then

$$\begin{aligned}
&P_s\begin{bmatrix} i_1 & \cdots & i_m & i_{m+1} & \cdots & i_{m+k} \\ j_1 & \cdots & j_m & 0 & \cdots & 0 \end{bmatrix} \\
&= \sum_{\left(\begin{smallmatrix} \alpha_1 & \cdots & \alpha_m \\ \beta_1 & \cdots & \beta_m \end{smallmatrix}\right)\in\left[\begin{smallmatrix} i_1 & \cdots & i_m \\ j_1 & \cdots & j_m \end{smallmatrix}\right]} \left(\frac{1}{(m+k)!}C_s^{\beta_1+\cdots+\beta_m}\frac{(\beta_1+\cdots+\beta_m)!}{\beta_1!\cdots\beta_m!}C^k_{m+k}\right. \\
&\quad\times\begin{Vmatrix} i_{m+1} & \cdots & i_{m+k} \\ 0 & \cdots & 0 \end{Vmatrix} - \sum_{q=1}^{m} P_s\begin{pmatrix} \alpha_1 & \cdots & \alpha_q \\ \beta_1 & \cdots & \beta_q \end{pmatrix} \\
&\quad\times\frac{1}{(m+k-q)!}C^{\beta_{q+1}+\cdots+\beta_m}_{s-(\alpha_1+\cdots+\alpha_q+\beta_1+\cdots+\beta_q)} \\
&\quad\left.\times\frac{(\beta_{q+1}+\cdots+\beta_m)!}{\beta_{q+1}!\cdots\beta_m!}C^k_{m+k-q}\begin{Vmatrix} i_{m+1} & \cdots & i_{m+k} \\ 0 & \cdots & 0 \end{Vmatrix}\right) \\
&= P_s\begin{bmatrix} i_1 & \cdots & i_m \\ j_1 & \cdots & j_m \end{bmatrix}\frac{1}{k!}\begin{Vmatrix} i_{m+1} & \cdots & i_{m+k} \\ 0 & \cdots & 0 \end{Vmatrix} \\
&\quad - P_s\begin{bmatrix} i_1 & \cdots & i_m \\ j_1 & \cdots & j_m \end{bmatrix} C^0_{s-(i_1+\cdots+i_m+j_1+\cdots+j_m)}\frac{1}{k!}\begin{Vmatrix} i_{m+1} & \cdots & i_{m+k} \\ 0 & \cdots & 0 \end{Vmatrix} \\
&= \begin{cases} 0 & \text{if } s \ge i_1+\cdots+i_m+j_1+\cdots+j_m, \\ P_s\begin{bmatrix} i_1 & \cdots & i_m \\ j_1 & \cdots & j_m \end{bmatrix}\frac{1}{k!}\begin{Vmatrix} i_{m+1} & \cdots & i_{m+k} \\ 0 & \cdots & 0 \end{Vmatrix} & \text{if } s < i_1+\cdots+i_m+j_1+\cdots+j_m. \end{cases}
\end{aligned}$$

□

4. Equations for Baker–Akhiezer function

Consider the KP hierarchy. This is a condition of compatibility of an infinite system of differential equations

$$\frac{\partial\psi}{\partial x_n} = L_n\psi, \tag{1}$$

where

$$L_n = \frac{\partial^n}{\partial x_1^n} + \sum_{i=2}^{n} B_n^i(x) \frac{\partial^{n-i}}{\partial^{n-i} x_1},$$

and ψ is a function of the type

$$\psi(x,k) = \exp\left(\sum_{j=1}^{\infty} x_j k^j\right)\left(1 + \sum_{i=1}^{\infty} \xi_i k^{-i}\right)$$

(here $k \in \mathbb{C}$ belongs to some neighbourhood of ∞ and $x = (x_1, x_2, \dots)$ is a finite sequence).

We use the notation

$$\partial_i = \frac{\partial}{\partial x_i}, \quad \partial = \partial_1.$$

A direct calculation gives the following lemma.

Lemma 2. *Conditions of compatibility of* (1) *are*

$$B_s^t = -\sum_{i=1}^{t-1} C_s^i \partial^i \xi_{t-i} - \sum_{j=2}^{t-1} B_s^j \sum_{i=0}^{t-j-1} C_{s-j}^i \partial^i \xi_{t-i-j}, \tag{2}$$

$$\partial_n \xi_i = \sum_{j=1}^{n+i-1} C_n^j \partial^j \xi_{i+n-j} + \sum_{k=2}^{n} B_n^k \sum_{j=0}^{n-k} C_{n-k}^j \partial^j \xi_{i+n-j-k}. \tag{3}$$

In this case, ψ is called a *Baker–Akhiezer function*.

Now consider the function

$$\ln \psi(x,k) = \sum_{j=1}^{\infty} x_j k^j + \sum_{j=1}^{\infty} \eta_j k^{-j},$$

where

$$\xi_j = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{i_1+\cdots+i_n=j} \eta_{i_1} \cdots \eta_{i_n}.$$

Lemma 3. *Let* $2 \le t \le s$. *Then*

$$B_s^t = -\sum_{n=1}^{\infty} \sum P_s \begin{pmatrix} i_1 & \cdots & i_n \\ j_1 & \cdots & j_n \end{pmatrix} \partial^{j_1} \eta_{i_1} \cdots \partial^{j_n} \eta_{i_n},$$

where the second sum is taken over all matrices

$$\begin{pmatrix} i_1 & \cdots & i_n \\ j_1 & \cdots & j_n \end{pmatrix}$$

such that $i_m \ge 1$, $j_m \ge 1$ *and* $i_1 + \cdots + i_n + j_1 + \cdots + j_n = t$.

Proof. By induction on t. For $t=2$ according to (2),

$$B_s^2 = -s\partial\xi_1 = -P_s\begin{pmatrix}1\\1\end{pmatrix}\partial\eta_1.$$

To prove the lemma for $t=N$ assume that it is proved for $t<N$. According to (2),

$$\begin{aligned}
B_s^t =& -\sum_{i=1}^{t-1} C_s^i\partial^i\left(\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{i_1+\cdots+i_n=t-i}\eta_{i_1}\cdots\eta_{i_n}\right)\\
&+\sum_{j=2}^{t-1}\left(\sum_{n=1}^{\infty}\sum_{i_1+\cdots+j_n=j}P_s\begin{pmatrix}i_1 & \cdots & i_n\\ j_1 & \cdots & j_n\end{pmatrix}\partial^{j_1}\eta_{i_1}\cdots\partial^{j_n}\eta_{i_n}\right)\\
&\times\left(\sum_{i=0}^{t-j-1}C_{s-j}^i\partial^j\left(\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{i_1+\cdots+i_n=t-i-j}\eta_{i_1}\cdots\eta_{i_n}\right)\right)\\
=& -\sum_{n=1}^{\infty}\sum\left(\frac{1}{n!}C_s^{j_1+\cdots+j_n}\frac{(j_1+\cdots+j_n)!}{j_1!\cdots j_n!}-\sum_{q=1}^{n-1}P_s\begin{pmatrix}i_1 & \cdots & i_q\\ j_1 & \cdots & j_q\end{pmatrix}\right.\\
&\left.\times\frac{1}{(n-q)!}C_{s-(i_1+\cdots+i_q+j_1+\cdots+j_q)}^{j_{q+1}\cdots j_n}\frac{(j_{q+1}+\cdots+j_n)!}{j_{q+1}!\cdots j_n!}\right)\partial^{j_1}\eta_{i_1}\cdots\partial^{j_n}\eta_{i_n}\\
=& -\sum_{n=1}^{\infty}\sum P_s\begin{pmatrix}i_1 & \cdots & i_n\\ j_1 & \cdots & j_n\end{pmatrix}\partial^{j_1}\eta_{i_1}\cdots\partial^{j_n}\eta_{i_n},
\end{aligned}$$

where the second sums are taken over all matrices

$$\begin{pmatrix}i_1 & \cdots & i_n\\ j_1 & \cdots & j_n\end{pmatrix}$$

such that $i_1+\cdots+i_n+j_1+\cdots+j_n=t$, $i_m\geq 1$, $j_m\geq 0$. According to Lemma 1, it is possible to consider the last sum with $j_m>0$ for all m. □

Lemma 4.

$$\partial_s\eta_r = \sum_{n=1}^{\infty}\sum P_s\begin{pmatrix}i_1 & \cdots & i_n\\ j_1 & \cdots & j_n\end{pmatrix}\partial^{j_1}\eta_{i_1}\cdots\partial^{j_n}\eta_{i_n},$$

where the second sum is taken by all matrices

$$\begin{pmatrix}i_1 & \cdots & i_n\\ j_1 & \cdots & j_n\end{pmatrix}$$

such that $i_m\geq 1$, $j_m\geq 1$ *and* $i_1+\cdots+i_n+j_1+\cdots+j_n=r+s$.

Proof. By induction on r. According to (3) and Lemma 3 for $r=1$

$$\begin{aligned}\partial_s\eta_1=\partial_s\xi_1&=\sum_{j=1}^{\infty}C_s^j\partial^j\xi_{s+1-j}+\sum_{k=2}^{s}B_s^k\sum_{j=0}^{s-k}C_{s-k}^j\partial^j\xi_{1+s-j-k}\\
&=\sum_{j=1}^{s}C_s^j\partial^j\xi_{s+1-j}-\sum_{k=2}^{\infty}\left(\sum_{n=1}^{\infty}\sum_{i_1+\cdots+j_n=k}P_s\begin{pmatrix}i_1&\cdots&i_n\\ j_1&\cdots&j_n\end{pmatrix}\partial^{j_1}\eta_{i_1}\cdots\partial^{j_n}\eta_{i_n}\right)\\
&\quad\times\left(\sum_{j=0}^{s-k}C_{s-k}^j\partial^j\xi_{1+s-j-k}\right)\\
&=\sum_{n=1}^{\infty}\sum P_s\begin{pmatrix}i_1&\cdots&i_n\\ j_1&\cdots&j_n\end{pmatrix}\partial^{j_1}\eta_{i_1}\cdots\partial^{j_n}\eta_{i_n},\end{aligned}$$

where the second sum in the last formula is taken over all matrices

$$\begin{pmatrix}i_1&\cdots&i_n\\ j_1&\cdots&j_n\end{pmatrix}$$

such that $i_1+\cdots+i_n+j_1+\cdots+j_n=s+1$, $i_m\geq 1$, $j_m\geq 0$. According to Lemma 1, in this sum it is sufficient to consider only matrices where $j_m>0$ for all m.

Now to prove the lemma for $r=N$, we assume that it is proved for $r<N$. According to (3)

$$\partial_s\left(\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{i_1+\cdots+i_n=r}\eta_{i_1}\cdots\eta_{i_n}\right)=\sum_{j=1}^{s+r-1}C_s^j\partial^j\xi_{s+r-j}+\sum_{k=2}^{\infty}B_s^k\sum_{j=0}^{s-k}C_{s-k}^j\partial^j\xi_{r+s-j-k}.$$

Thus according to Lemma 3, Lemma 1 and inductive hypothesis,

$$\begin{aligned}\partial_s\eta_r&=\sum_{j=1}^{\infty}C_s^j\partial^j\left(\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{i_1+\cdots+i_n=s+r-j}\eta_{i_1}\cdots\eta_{i_n}\right)\\
&\quad+\sum_{k=2}^{\infty}\left(\sum_{i_1+\cdots+j_n=k}P_s\begin{pmatrix}i_1&\cdots&i_n\\ j_1&\cdots&j_n\end{pmatrix}\partial^{j_1}\eta_{i_1}\cdots\partial^{j_n}\eta_{i_n}\right)\sum_{j=0}^{s-k}C_{s-k}^j\partial^j\\
&\quad\times\left(\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{i_1+\cdots+i_n=r+s-j-k}\eta_{i_1}\cdots\eta_{i_n}\right)-\partial_s\left(\sum_{n=2}^{\infty}\frac{1}{n!}\sum_{i_1+\cdots+i_n=r}\eta_{i_1}\cdots\eta_{i_n}\right)\\
&=\sum_{n=1}^{\infty}\sum P_s\begin{pmatrix}i_1&\cdots&i_n\\ j_1&\cdots&j_n\end{pmatrix}\partial^{j_1}\eta_{i_1}\cdots\partial^{j_n}\eta_{i_n}\end{aligned}$$

where the second sum in the last formula is taken over all matrices

$$\begin{pmatrix}i_1&\cdots&i_n\\ j_1&\cdots&j_n\end{pmatrix}$$

such that $i_1+\cdots+i_n+j_1+\cdots+j_n=s+r$, $i_m\geq 1$, $j_m\geq 1$. □

5. KP hierarchy

According to [3], the Baker–Akhiezer function ψ is

$$\psi(x,k)=\exp\left(\sum x_j k^j\right)\frac{\tau(x_1-k^{-1},x_2-\frac{1}{2}k^{-2},x_3-\frac{1}{3}k^{-3},\dots)}{\tau(x_1,x_2,x_3,\dots)}$$

for some function $\tau(x_1,x_2,\dots)$. This gives a possibility to describe the KP hierarchy as an infinite system of differential equations on $v(x,k)=-\ln\tau(x,k)$ (see [7]). Indeed,

$$\sum_{j=1}^{\infty}\eta_j k^{-j}=\ln\psi(x,k)-\sum_{j=1}^{\infty}x_j k^j=-v\left(x_1-k^{-1},x_2-\frac{1}{2}k^{-2},\dots\right)+v(x)$$
$$=\sum_{n=1}^{\infty}\sum_{i_1+\cdots+i_n=j}\frac{(-1)^{n+1}}{n!i_1\cdots i_n}\partial_{i_1}\cdots\partial_{i_n}v(x)k^{-j}.$$

Therefore,

$$\eta_r=\sum_{n=1}^{\infty}\sum_{i_1+\cdots+i_n=r}\frac{(-1)^{n+1}}{n!i_1\cdots i_n}\partial_{i_1}\cdots\partial_{i_n}v. \tag{4}$$

Theorem 2. *There exist universal rational coefficients*

$$R_r\begin{pmatrix}s_1&\cdots&s_n\\t_1&\cdots&t_n\end{pmatrix},\qquad R_{ij}\begin{pmatrix}s_1&\cdots&s_n\\t_1&\cdots&t_n\end{pmatrix}$$

such that

$$\eta_r=\frac{1}{r}\partial_r v+\sum_{n=1}^{\infty}\sum R_r\begin{pmatrix}s_1&\cdots&s_n\\t_1&\cdots&t_n\end{pmatrix}\partial_{s_1}\partial^{t_1}v\cdots\partial_{s_n}\partial^{t_n}v, \tag{5}$$

$$\partial_i\partial_j v=\sum_{n=1}^{\infty}\sum R_{ij}\begin{pmatrix}s_1&\cdots&s_n\\t_1&\cdots&t_n\end{pmatrix}\partial_{s_1}\partial^{t_1}v\cdots\partial_{s_n}\partial^{t_n}v, \tag{6}$$

where the second sums are taken over all matrices

$$\begin{pmatrix}s_1&\cdots&s_n\\t_1&\cdots&t_n\end{pmatrix}$$

such that $s_m,t_m\geq 1$, *and the sum* $s_1+\cdots+s_n+t_1+\cdots+t_n$ *is equal to* r *for* (5) *and* $i+j$ *for* (6). *Moreover,*

$$R_{ij}\begin{pmatrix}s_1&\cdots&s_n\\1&\cdots&1\end{pmatrix}=\frac{j}{s_1\cdots s_n}P_i(s_1\cdots s_n).$$

Proof. By induction on k and $i+j$. For $i+j=2$, the theorem is obvious. For $r=1$, it follows from (4). To prove the theorem for $i+j=N$ and $r=N-1$, assume that it is proved for $i+j<N$ and $r<N-1$. Later we consider that $s_m, t_m \geq 1$ and $\sigma_n = s_1+\cdots+s_n+t_1+\cdots+t_n$. Then according to (4) and (6)

$$\eta_r = \frac{1}{r}\partial_r v + \sum_{n=2}^{\infty}\sum_{s_1+\cdots+s_n=r}\frac{(-1)^{n+1}}{n!s_1\cdots s_n}\partial_{s_1}\cdots\partial_{s_n}v(x)$$
$$= \frac{1}{r}\partial_r v + \sum_{n=1}^{\infty}\sum_{\sigma_n=r} R_r\begin{pmatrix} s_1 & \cdots & s_n \\ t_1 & \cdots & t_n\end{pmatrix}\partial_{s_1}\partial^{t_1}v\cdots\partial_{s_n}\partial^{t_n}v.$$

Thus according to (5) and (6) and Lemma 4

$$\frac{1}{j}\partial_i\partial_j v = \partial_i\eta_j - \partial_i\left(\sum_{n=1}^{\infty}\sum_{\sigma_n=j}R_j\begin{pmatrix} s_1 & \cdots & s_n \\ t_1 & \cdots & t_n\end{pmatrix}\partial_{s_1}\partial^{t_1}v\cdots\partial_{s_n}\partial^{t_n}v\right)$$
$$= \sum_{n=1}^{\infty}\sum_{\sigma_n=i+j}P_i\begin{pmatrix} s_1 & \cdots & s_n \\ t_1 & \cdots & t_n\end{pmatrix}\partial^{t_1}\eta_{s_1}\cdots\partial^{t_n}\eta_{s_n}$$
$$-\partial_i\left(\sum_{n=1}^{\infty}\sum_{\sigma_n=j}R_j\begin{pmatrix} s_1 & \cdots & s_n \\ t_1 & \cdots & t_n\end{pmatrix}\partial^{t_1}\partial_{s_1}v\cdots\partial^{t_n}\partial_{s_n}v\right)$$
$$= \sum_{n=1}^{\infty}\sum_{s_1+\cdots+s_n+n=i+j}P_i\begin{pmatrix} s_1 & \cdots & s_n \\ 1 & \cdots & 1\end{pmatrix}\partial\left(\frac{1}{s_1}\partial_{s_1}v\right)\cdots\partial\left(\frac{1}{s_n}\partial_{s_n}v\right)$$
$$+\sum_{n=1}^{\infty}\sum_{\sigma_n=i+j,t_1+\cdots+t_n>n}R_{ij}\begin{pmatrix} s_1 & \cdots & s_n \\ t_1 & \cdots & t_n\end{pmatrix}\partial_{s_1}\partial^{t_1}v\cdots\partial_{s_n}\partial^{t_n}v.$$

The obvious relation

$$P_i\begin{pmatrix} s_1 & \cdots & s_n \\ 1 & \cdots & 1\end{pmatrix} = P_i(s_1\cdots s_n)$$

concludes the proof. □

Consequence 1. The system (6) is equivalent to KP hierarchy.

Remark. The first equation from (6) is the "integrated over x_1" KP equation

$$\partial_2^2 v = \tfrac{4}{3}\partial_3\partial_1 v - \tfrac{1}{3}\partial_1^4 v + 2(\partial_1^2 v)^2.$$

Relation (6) was first deduced in [4] from the Hirota hierarchy [3]. This method produces other formulas for coefficients R_{ij}. The equality of coefficients provides non-trivial combinatorial identities.

Remark. From the consequence, it follows that any formal solution of KP hierarchy is defined up to a constant by an arbitrary infinite set of series of one variable $f_i(x_1) = \partial_i v|_{x_2=x_3=\cdots=0}(i=1,2,\dots)$.

6. Gelfand–Dikii hierarchy

According to [8], the set of solution of n-Gelfand–Dikii (n-KdV) hierarchy bijectively corresponds to the set of solutions of KP hierarchy independent from x_n. In this case according to Theorem 2

$$0=\partial_m\partial_n v=\frac{mn}{m+n-1}\partial_{n+m-1}\partial v+\sum_{m=1}^{\infty}\sum R_{mn}\begin{pmatrix} s_1 & \cdots & s_m \\ t_1 & \cdots & t_m \end{pmatrix}\partial_{s_1}\partial^{t_1}v\cdots\partial_{s_m}\partial^{t_m}v, \tag{7}$$

where $1\le s_j\le n+m-2$, $t_j\ge 1$. This gives recurrence formulas expressing $\partial_k\partial v$ for $k>n$ via $\partial_r\partial v$ for $r<n$. Thus, we have relations

$$\partial\partial_{n+r}v=\sum_{m=1}^{\infty}\sum N^m_{1(n+1)}\begin{pmatrix} s_1 & \cdots & s_m \\ t_1 & \cdots & t_m \end{pmatrix}\partial_{s_1}\partial^{t_1}v\cdots\partial_{s_m}\partial^{t_m}v, \tag{8}$$

where $t_j\ge 1$, $s_j<n$, $\sum_{j=1}^{m}(s_j+t_j)=n+r+1$.

Example. For $n=2$, system (8) is reduced to the K∂V hierarchy.

Comparing the systems (6) and (8), we find the following theorem.

Theorem 3. *The n-Gelfand–Dikii hierarchy is equivalent to a system of differential equations of the form*

$$\partial_i\partial_j v=\sum_{m=1}^{\infty}\sum N^m_{i_j}\begin{pmatrix} s_1 & \cdots & s_m \\ t_1 & \cdots & t_m \end{pmatrix}\partial_{s_1}\partial^{t_1}v\cdots\partial_{s_m}\partial^{t_m}v, \tag{9}$$

where $i,j\ge 1$, $1\le s_\alpha\le n-1$, $t_\alpha\ge 1$, $\sum_{i=1}^{m}(s_\alpha+t_\alpha)=i+j$ *and*

$$N^m_{ij}\begin{pmatrix} s_1 & \cdots & s_m \\ t_1 & \cdots & t_m \end{pmatrix}$$

are some universal rational coefficients.

Example.

1. For $n=3$, the first equation from system (9) is the Boussinesq equation

$$\partial_2^2 v=-\tfrac{1}{3}\partial^4 v+2(\partial^2 v)^2.$$

2. For $n=4$, the first equations of the system (9) are

$$\partial_2^2 v=\tfrac{4}{3}\partial_3\partial v-\tfrac{1}{3}\partial^4 v+2(\partial^2 v)^2,\qquad \partial_3\partial_2 v=-\tfrac{3}{2}\partial_2\partial^3 v+3\partial_2\partial v\partial^2 v,$$
$$\partial_3^2 v=-\tfrac{1}{4}\partial_3\partial^3 v+\tfrac{1}{8}\partial^6 v+\tfrac{9}{8}(\partial_2\partial v)^2-\tfrac{9}{8}(\partial^3 v)^2-\tfrac{9}{4}\partial^4 v\partial^2 v+3(\partial^2 v)^3.$$

Remark. The structure of system (9) demonstrates that any formal solutions of n-Gelfand–Dikii equation is defined up to a constant by an arbitrary set of $n-1$ series of one variable $f_i(x_1) = \partial_i v|_{x_2=x_3=\cdots=0} (i = 1, \dots, n-1)$.

According to [6], the dispersionless (quasi-classical) limit of n-Gelfand–Dikii hierarchy is given by n-Gelfand–Dikii hierarchy without all monomials

$$R_{ij}\begin{pmatrix} s_1 & \cdots & s_n \\ t_1 & \cdots & t_n \end{pmatrix} \partial_{s_1}\partial^{t_1} v \cdots \partial_{s_n}\partial^{t_n} v,$$

where $\sum_{i=1}^{n} t_j > n$.

Thus, from Theorem 2 we get the following consequence.

Consequence 2. Dispersionless (quasi-classical) limit of n-Gelfand–Dikii hierarchy is given by the infinite system

$$\frac{1}{j}\partial_i\partial_j v = \sum_{m=1}^{\infty} \sum_{s_1+\cdots+s_m=i+j-m} P_i(s_1\cdots s_m)\frac{1}{s_1}\partial\partial_{s_1} v \cdots \frac{1}{s_m}\partial\partial_{s_m} v, \tag{10}$$

where $i, j = 1, 2, \dots,$ and the equation $\partial_n v = 0$.

7. Proof of Theorem 1

According to [6], one of the A_{n-1}-potentials is

$$\tilde{F}(x_1, \dots, x_{n-1}) = v\left(x_1, \frac{x_2}{2}, \dots, \frac{x_{n-1}}{n-1}, 0, \dots\right),$$

where v satisfies the dispersionless n-Gelfand–Dikii hierarchy.

From Consequence 2, it follows that

$$\begin{aligned} 0 = \partial_\ell\partial_n\tilde{F} &= \sum_{m=1}^{\infty} \sum_{s_1+\cdots+s_m=\ell+n-m} P_n(s_1\cdots s_m)\partial\partial_{s_1}\tilde{F}\cdots\partial\partial_{s_m}\tilde{F} \\ &= n\partial\partial_{n+\ell-1}\tilde{F} + \sum_{m=2}^{\infty} \sum_{s_1+\cdots+s_m=\ell+n-m} P_n(s_1\cdots s_m)\partial\partial_{s_1}\tilde{F}\cdots\partial\partial_{s_m}\tilde{F}. \end{aligned}$$

Thus,

$$\partial\partial_{n+\ell-1}\tilde{F} = -\frac{1}{n}\sum_{m=2}^{\infty} \sum_{s_1+\cdots+s_m=\ell+n-m} P_n(s_1\cdots s_m)\partial\partial_{s_1}\tilde{F}\cdots\partial\partial_{s_m}\tilde{F}. \tag{11}$$

According to [10],

$$\partial\partial_s\tilde{F} = x_{n-s}, \qquad \partial\partial_{n+s}\tilde{F} = -s\partial_s\tilde{F}$$

for $s < n$. Thus, because $\partial\partial_n \tilde{F} = 0$ and according to (11) we have $\partial\partial_{n+\ell-1}\tilde{F} = x_{1-\ell}$ for all ℓ. Therefore,

$$\partial_s \tilde{F} = -\frac{1}{s} x_{-s} = \partial_s F_A,$$

and $F_A - \tilde{F}$ is a linear form. According to [2, p. 16], this form is equal to 0. According to [11], we have $F_B(x_1, \dots, x_n) = F_A(x_1, 0, x_2, 0, \dots, x_{n-1}, 0, x_n)$.

These are the potentials found from our formulas

$$(A_2) \quad F = \tfrac{1}{2}x_1^2 x_2 + \tfrac{1}{24}x_2^4,$$

$$(A_3) \quad F = \tfrac{1}{2}x_1^2 x_3 + \tfrac{1}{2}x_1 x_2^2 + \tfrac{1}{4}x_2^2 x_3^2 + \tfrac{1}{60}x_3^5,$$

$$(A_4) \quad F = \tfrac{1}{2}x_1^2 x_4 + x_1 x_2 x_3 + \tfrac{1}{6}x_2^3 + \tfrac{1}{12}x_3^4 + \tfrac{1}{2}x_2 x_3^2 x_4 + \tfrac{1}{4}x_2^2 x_4^2 + \tfrac{1}{6}x_3^2 x_4^3 + \tfrac{1}{120}x_4^6,$$

$$(A_5) \quad F = \tfrac{1}{2}x_1^2 x_5 + \tfrac{1}{2}x_2^2 x_3 + \tfrac{1}{4}x_2^2 x_5^2 + x_2 x_3 x_4 x_5 + x_1 x_2 x_4 + \tfrac{1}{6}x_2 x_4^3 + \tfrac{1}{2}x_1 x_3^2 + \tfrac{1}{2}x_3^2 x_4^2 + \tfrac{1}{6}x_3^3 x_5 + \tfrac{1}{6}x_3^2 x_5^3 + \tfrac{1}{2}x_3 x_4^2 x_5^2 + \tfrac{1}{6}x_4^4 x_5 + \tfrac{1}{8}x_4^2 x_5^4 + \tfrac{1}{210}x_5^7,$$

$$(B_2) \quad F = \tfrac{1}{2}x_1^2 x_2 + \tfrac{1}{60}x_2^5,$$

$$(B_3) \quad F = \tfrac{1}{2}x_1^2 x_3 + \tfrac{1}{2}x_1 x_2^2 + \tfrac{1}{6}x_2^3 x_3 + \tfrac{1}{6}x_2^2 x_3^3 + \tfrac{1}{210}x_3^7.$$

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